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On The Diophantine Equation $x^3 + y^3 = z^2$

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ABSTRACT

In this article, we proved that an integral solution (*a, b, c*) to the Diophantine equation $x^3 + y^3 = z^2$ is of the $a = rs$, $b = rt$, $c = (r^3 m)^{1/2}$ for any two positive integers *s, t.*

Keywords: Integral solution; Diophantine equation

INTRODUCTION

The expression 'Diophantine equation' comes from Diophantus of Alexandria (about A.D. 250). There are many cases of the Diophantine equation have been studied over the years. Arif and Abu Muriefah (1998) proved that when *m* be odd, the Diophantine equation $x^2 + 3^m = y^n$, $n > 3$ has only one solution in positive integers *x, y, m* and the unique solution is given by $m = 5 + 6M$, $x=10.3^{3M}$, $y = 7.3^{2M}$ and $n = 3$. Abu Mureifah and Bugeaud (2006) came out with their survey on recent results on the Diophantine equation $x^2 + c = y^n$. While, Manley (2006) considered the Diophantine equation $x^4 + py^4 = z^4$ where *p* is prime and $p \not\equiv 3 (\text{mod} 8)$ and shown that all quadratic solutions are inherited. That is, all quadratic solutions can be easily obtained from integer solutions to be simpler equation $x^4 + py^4 = z^2$.

Terai (1993) conjectured that if $a^2 + b^2 = c^2$ with (*a, b, c)* = 1 and *a* even, then the equation $x^2 + b^m = c^n$ has the only positive integral solution $(x,m,n) = (a,2,2)$. Using Baker's efficient method, Le (1995) proved that Terai's conjecture holds if $b>8.10^6$, $b \equiv \pm 5 (\text{mod} 8)$and *c* is a prime power. Then, in a paper by Yuan and Wang (1998), by a completely different method, they proved that if $a^2 + b^2 = c^2$, (*a, b, c*) = 1 , $b \equiv \pm 5 (\text{mod} 8)$and *c* is a prime, then Terai's Conjecture holds.

Many special cases of the equation $x^2 + y^n = z^n$ have been considered over the years. For instance, Alkhazin, Alkhawarizmi, Abu Kamil, Fibonanci and etc. However, on June 23, 1993 at the Isaac Newton Institute of Cambridge (England), Professor Andrew Wiles (Princeton University) made a striking announcement. He had found a proof of Fermat's Last Theorem which stated that "Let *n* be an integer greater than or equal to 3. Then there are no nonzero integers A, B, C such that $A^n + B^n = C^n$."

In article by Bruin (2005), he considered a special instance of the equation $x^3 + y^9 = z^2$. He deal with the one remaining case and proved the theorem which stated that "The primitive solutions to the equation are $(x,y,z) = \{(\pm 1, \pm 1, 0), (0, 1, \pm 1), (1, 0, \pm 1), (2, 1, \pm 3), (-7, 2, \pm 13)\}$."

Levesque (2003), in his survey article mentioned that Zagier first showed that all integral solutions of $x^3 + y^3 = z^2$ are given by the following parameterizations:

$$X = s^4 + 6s^2t^2 - 3t^4,\ Y = -s^4 + 6s^2t^2 + 3t^4,\ Z = 6st(s^4 + 3t^4);$$

$$X = s^4 + 8st^3, \qquad Y = -4s^3t + 4t^4, \qquad Z = s^6 - 20s^3t^3 - 8t^6;$$

$$X = \frac{s^4 + 6s^2t^2 - 3t^4}{4}, Y = \frac{-s^4 + 6s^2t^2 + 3t^4}{4}, Z = \frac{3st(s^4 + 3t^4)}{4}.$$

The above results prompt us to study the Diophantine equation of the form $x^3 + y^3 = z^2$ and determine its integral solutions. In this paper, we will attempt to give another form of solution to the result given by Zagier as reported in Levesque (2003).

Integral Solution to the Diophantine Equation $x^3 + y^3 = z^2$

Theorem 2.1 will give the integral solution to the equation $x^3 + y^3 = z^2$.

Theorem 2.1

Suppose *a, b*, and *c* are positive integers and *r* be a common factor of *a* and *b*. Suppose (a, b, c) is an integral solution to the equation $x^3 + y^3 = z^2$, then there exist positive integers *s, t* and *m* such that

$$a = rs,\ \ b = rt,\ \ c = (r^3m)^{1/2}.$$

Proof

Suppose (a, b, c) is an integral solution to the equation $x^3 + y^3 = z^2$ such that $a^3 + b^3 = c^2$

Since *r* is a common factor of *a* and *b*, there exist integers *s, t* such that

$$a = rs,\ \ b = rt$$

Since $r \mid a$ and $r \mid b$, then $\frac{c^2}{r^3}$ is an integer. Thus, there exists integer such that

$$c^2 = r^3m.$$

That is $c = (r^3m)^{1/2}$.

Hence, there exist integers $s > 0$, $t > 0$ and such that

$$a = rs,\ \ b = rt,\ \ c = (r^3m)^{1/2n}.\square$$

Corollary 2.1 give the general integral solution to the Diophantine equation $x^3 + y^3 = z^2$.

Corollary 2.1

Let *m* be any integer and $v > 0$. Suppose (a, b, c) integers such that $a^3 + b^3 = c^2$.Then

$$x = am^{2v},\ \ y = bm^{2v},\ \ z = cm^{3v}$$

integral solution to the equation $x^3 + y^3 = z^2$.

Proof

We have $a^3 + b^3 = c^2$. (2.1)

Multiplying equation (2.1) by m^{3v}, we have

$$(am^v)^3 + (bm^v)^3 = m^{nv}(cm^v)^2. \quad (2.2)$$

Multiplying equation (2.2) by m^{3v}, will give

$$(am^{2v})^3 + (bm^{2v})^3 = (cm^{3v})^2..$$

Thus,

$$x = am^{2v}, \quad y = bm^{2v}, \quad z = cm^{3v}$$

integral solution to the equation $x^3 + y^3 = z^2$. □

AN ALTERNATIVE METHOD OF INTEGRAL SOLUTION TO THE DIOPHANTINE EQUATION $x^3 + y^3 = z^2$

Theorem 3.1 will give an alternative method of finding the integral solution to the equation $x^3 + y^3 = z^2$. First, we have the following assertion.

Lemma 3.1

Let n be an integer. Then there exist integers $m > 0$, $u > 0$ and $v > 0$ such that $u^{2m-1}n = v^{2m}$.

Proof

Let $n = \prod_{i=1}^{k} p_i^{e_i}$ be the prime power decomposition of n.

Consider the case when e_i can be both odd or even number.

Suppose $e_i = 2\, s_i$ for all $i = 2, \ldots, r$ and $e_i = 2s_i + 1$ for all $i = r + 1, r + 2, \ldots, k$ where s_i postive integer.

After the rearrangement of powers of p in the prime power decomposition of n we obtain

$$n = \prod_{i=1}^{r} p_i^{2s_i} \prod_{i=r+1}^{k} p_i^{2s_i+1} = \prod_{i=1}^{r} p_i^{2s_i} \prod_{i=r+1}^{k} p_i^{2s_i} \prod_{i=r+1}^{k} p_i.$$

Suppose $\gcd(s_1, s_2, \ldots, s_k) = d$. Then $s \mid s_i$, .

Thus, $s_i = dt_i$ for some t_i.

That is,

$$n = \prod_{i=1}^{r} p_i^{2dt_i} \prod_{i=r+1}^{k} p_i^{2dt_i} \prod_{i=r+1}^{k} p_i = \left(\prod_{i=1}^{r} p_i^{t_i} \prod_{i=r+1}^{k} p_i^{t_i} \right)^{2d} \prod_{i=r+1}^{k} p_i.$$

Let $d = m$, then there exists integer $m > 0$ such that

$$n = \left(\prod_{i=1}^{r} p_i^{t_i} \prod_{i=r+1}^{k} p_i^{t_i}\right)^{2m} \prod_{i=r+1}^{k} p_i. \tag{3.1}$$

Let $w = \prod_{i=1}^{r} p_i^{t_i} \prod_{i=r+1}^{k} p_i^{t_i}$; $u = \prod_{i=r+1}^{k} p_i$.

By the equation (3.1), we obtain $n = w^{2m} u$

Thus, $u^{2m-1} n = w^{2m} u^{2m} = (wu)^{2m}$.
Let $v = wu$. Hence, $u^{2m-1} n = v^{2m}$. □

Theorem 3.1.
Suppose s, t any positive integers such that $s^3 + t^3 = m$. Then there exists positive integer r such that

$$x = rs, \quad y = rt, \quad z = rv$$

where $v = (rm)^{1/2}$ integral solution to the equation $x^3 + y^3 = z^2$.

Proof.
Suppose s, t any positive integers such that $s^3 + t^3 = m$.. (3.2)

From Lemma 3.1, there exists integer r such that $rm = v^2$
Multiplying equation (3.2) by r, we have

$$r(s^3 + t^3) = rm = v^2. \tag{3.3}$$

Multiplying equation (3.3) by r^2, we obtain

$$r^3(s^3 + t^3) = r^2 v^2 = (rv)^2.$$

That is, $(rs)^3 + (rt)^3 = (rv)^2$.

Thus,

$$x = rs, \quad y = rt, \quad z = rv$$

Integral solution to the equation $x^3 + y^3 = z^2$. □

CONCLUSION

From our investigation, we found that for a, b and c positive integers and r a common factor of a and b, the integral solution (a, b, c) to the Diophantine equation $x^3 + y^3 = z^2$ is of the form $a = rs, \; b = rt, \; c = (r^3 m)^{1/2}$ for any positive integers s, t. Indirectly, we give another form of the Zagier's result for the Diophantine equation $x^3 + y^3 = z^2$.

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